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TECHNICAL NOTE

Solution for non-Fourier dual phase lag heat conduction in a semi-infinite slab with surface heat flux

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1. INTRODUCTION

1.1. *Scope*

This work derives a solution for transient temperature in a semi-infinite slab subjected to constant heat flux at its surface, while governed by the non-Fourier "dual phase lag" (DPL) model [1, 2] of heat conduction developed recently. The solution shows temperatures predicted with the DPL model can differ significantly from predictions based on the classical model of Fourier's Law.

Interest in the DPL model should grow in the near future because it shows good agreement with experiments across a wide range of length and time scales [3], including the "microscale" range of increasing importance. Hence, the solution derived here will permit quick estimates of DPL behavior for practical situations, such as laser heating of semiconductors during fabrication of microscale electronic devices. Also, this solution will help test numerical solution methods likely to be developed for the model.

The first part of this work uses the DPL model to derive a solution for transient temperature in a semi-infinite slab with constant surface temperature. In the second part, this solution serves as the starting point for deriving the desired solution for constant surface heat flux. Also, the solution for constant surface temperature derived in the first part is a convenient alternative to others published previously.

Reference [1] also uses the DPL model to derive a solution for a semi-infinite slab with constant surface temperature, using an approach similar to that taken here. However, the solution in [1] is not adopted here, as discussed in the Appendix.

1.2. *The DPL model*

The DPL model accounts for the non-zero times required by heat flux and temperature gradient to gradually become established in response to thermal disturbances (e.g. an imposed heat flux). In contrast, Fourier's Law assumes heat flux and temperature gradient become established immediately (response time of zero). Although Fourier's Law is usually very accurate, modeling this gradual response is essential for accurate predictions of transient temperature during, for example, microtime ($<10^{-12}$ s) laser heating of metal thin films ($\leqslant 1\ \mu$) [3]. Here, the gradual response occupies the time of laser heating. Consequently, predictions based on Fourier's Law do not capture the behavior of temperature corresponding to the gradual response. Modeling this gradual response can also be important on the "macroscale" where length and time scales are relatively large, as in transient heating of sand [1] involving scales on the orders of 0.01 m and 1.0 s.

The one-dimensional DPL model [2] relating heat flux to temperature gradient is

$$q+\tau_q\frac{\partial q}{\partial t} = -k\frac{\partial T}{\partial x}-k\tau_T\frac{\partial^2 T}{\partial t\,\partial x} \qquad (1)$$

where the "thermal lags" (delays) τ_q and τ_T are approximately the times needed for gradual response of heat flux and temperature gradient, respectively. Further, $\partial q/\partial t$ and $\partial^2 T/\partial t\,\partial x$ represent transient behavior of heat flux and temperature gradient during this response. After the gradual response is complete, heat flux and temperature gradient achieve the values given by Fourier's Law.

In general terms, τ_q and τ_T are interpreted as non-zero times accounting for the effects of "thermal inertia" and "microstructural interaction," respectively [3]. Hence, τ_q is the delay in establishing heat flux and associated conduction through a solid. This delay tends to induce thermal waves with sharp wave-fronts separating heated and unheated zones in the solid. However, τ_T is the delay in establishing the temperature gradient across the solid during which conduction occurs through its small-scale structures. Thus, τ_T smooths the sharp wave-fronts of τ_q by promoting conduction, resulting in non-Fourier diffusion-like conduction. References [1] and [3] provide specific interpretations of the lags for several situations and conditions of applicability for the DPL model.

Values for thermal lags are typically very small for continuous solids, implying the immediate response of Fourier's Law is usually an accurate approximation except for very small periods of time after thermal disturbances. For example, with gold [2] $\tau_q \approx 0.7\times10^{-12}$ s and $\tau_T \approx 89.0\times10^{-12}$ s. However, for non-continuous solids the lags can be relatively large, shown by $\tau_q = 8.9$ s and $\tau_T = 4.5$ s for one type of sand [1].

Equation (1) reduces to the non-Fourier "Cattaneo Vernotte" model underlying hyperbolic heat conduction [4, 5] by setting $\tau_T = 0$. Further, setting $\tau_q = \tau_T = 0$ (immediate response) reduces the equation to Fourier's Law. The equation also reduces to Fourier's Law for steady state conditions even with non-zero τ_q and τ_T. In addition to equation (1), there are higher order DPL models involving, for example, τ_q^2 [1, 3]. These higher-order models are not considered here.

The one-dimensional energy equation for an incompressible solid with no internal heat generation or absorption is

$$\rho C\frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x}. \qquad (2)$$

Combining equations (1) and (2), and treating ρ, C, k and thermal lags as constants while eliminating q, leads to the

NOMENCLATURE

a, A	parameters defined with equation (6a)
b	parameter defined with equation (7a)
B	ratio of thermal lags $= (1/2)(\tau_T/\tau_q)$
c_w	speed of thermal wave for hyperbolic conduction
C	specific heat
I_1, I_2	integrands defined by equations (6a) and (7a), respectively
I_3, I_4	integrands defined by equations (11a) and (12a), respectively
k	thermal conductivity
p	parameter for Laplace transform
q	heat flux
q_0	heat flux at $x = 0$
t	time
T	temperature
T_i	initial temperature
T_o	temperature at $x = 0$
u	dummy integration variable in equation (10)
x	location
x_w	location of thermal wave-front for hyperbolic conduction.

Greek symbols

α	thermal diffusivity $= k/(\rho C)$
β	dimensionless time $= t/(2\tau_q)$
δ	dimensionless location $= x/[2(\alpha\tau_q)^{1/2}]$
δ_w	dimensionless location of thermal wave-front for hyperbolic conduction
θ	dimensionless temperature for slab with constant surface temperature $= (T-T_i)/(T_o-T_i)$
$\bar{\theta}$	dimensionless temperature in Fourier space
$\tilde{\bar{\theta}}$	dimensionless temperature in Fourier/Laplace space
ξ	parameter for Fourier sine transform
ξ_1, ξ_2	parameters defined by equations (8a) and (8b), respectively
ρ	density
τ_q, τ_T	thermal lags defined with equation (1)
φ	dimensionless temperature for slab with constant surface heat flux $= (T-T_i)k/[q_0(\alpha\tau_q)^{1/2}]$
φ_{jump}	jump in surface temperature for hyperbolic conduction
ψ	dimensionless heat flux $= q(x, t)/q_o$.

one-dimensional DPL heat conduction equation for temperature $T(x, t)$:

$$\tau_q \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + \alpha\tau_T \frac{\partial^3 T}{\partial t\,\partial x^2} \tag{3}$$

with thermal diffusivity α. In equation (3), $\partial^2 T/\partial t^2$ represents thermal waves induced by τ_q while $\partial^3 T/\partial t\,\partial x^2$ represents smoothing of the waves by τ_T.

Alternatively, combining equations (1) and (2) while eliminating T gives the analog to equation (3) for one-dimensional heat flux where $q(x, t)$ replaces T in equation (3). This analog is used later for the slab with constant surface heat flux.

Solutions to DPL problems converge to corresponding Fourier (classical diffusion) solutions when enough time elapses after thermal disturbances end. ("Corresponding" means all aspects of DPL and Fourier problems are identical except for their different conduction models.) At short times, however, DPL solutions show non-Fourier diffusion-like behavior [6].

Equation (3) reduces to the hyperbolic heat equation [4, 5] for $\tau_T = 0$ and the Fourier heat equation for $\tau_q = \tau_T = 0$. For steady state conditions, equation (3) reduces to the Fourier equation even for non-zero τ_q and τ_T. Further, solutions to problems governed by equation (3) are equivalent to corresponding Fourier solutions when $\tau_q = \tau_T$ and initial temperatures are not changing with time [2]. Also, equation (3) is equivalent [2] to a "two step" model [7] for microtime heating of metals. Hence, solutions to DPL problems posed with equation (3) can be solutions to problems involving the "two step" model. Finally, equation (3) is analogous to an equation for momentum transport in a viscoelastic (non-Newtonian) liquid. Thus, solutions to some viscoelastic problems can be adopted for DPL problems, as described later.

2. CONSTANT SURFACE TEMPERATURE

2.1. *Formulation*

The semi-infinite slab occupies the half-space $x > 0$ and is initially at temperature T_i. At $t = 0^+$ the surface along $x = 0$ is suddenly raised to constant temperature T_o. With the formulation of [6], equation (3) for the transient temperature in the slab becomes

$$\frac{\partial^2 \theta}{\partial \beta^2} + 2\frac{\partial \theta}{\partial \beta} = \frac{\partial^2 \theta}{\partial \delta^2} + B\frac{\partial^3 \theta}{\partial \beta\,\partial \delta^2} \tag{4}$$

with dimensionless temperature $\theta(\delta, \beta) = (T-T_i)/(T_o-T_i)$, time $\beta = t/(2\tau_q)$ and location $\delta = x/[2(\alpha\tau_q)^{1/2}]$. Also, $B = \frac{1}{2}(\tau_T/\tau_q)$ is the dimensionless ratio of thermal lags where $B = 0$ and $B = 1/2$ correspond to hyperbolic and Fourier conduction, respectively. Non-zero values of B cited thus far [3] range from approximately 5×10^{-4} for liquid helium to 100 for metals.

The initial and boundary conditions for the slab are

$$\theta(\delta, 0) = \frac{\partial \theta(\delta, 0)}{\partial \beta} = 0 \tag{4a,b}$$

$$\theta(0, \beta) = H(\beta) \tag{4c}$$

$$\theta(\delta \to \infty, \beta) \to 0 \tag{4d}$$

where $H(\beta)$ is the unit step function.

2.2. *Solution*

The problem of equations (4)–(4d) is solved here by first taking the Fourier sine transform of each term in the equations to eliminate δ-derivatives. This transform reduces equation (4) to an ordinary differential equation involving only β-derivatives. The definition of this transform with parameter ξ is

$$\bar{\theta}(\xi, \beta) = (2/\pi)^{1/2} \int_0^\infty \theta(\delta, \beta) \sin(\xi\delta)\,d\delta \tag{5a}$$

with inverse transform

$$\theta(\delta, \beta) = (2/\pi)^{1/2} \int_0^\infty \bar{\theta}(\xi, \beta) \sin(\xi\delta)\,d\xi. \tag{5b}$$

Next, using the Laplace transform to eliminate β-derivatives in the ordinary differential equation results in an algebraic expression for the double-transformed temperature $\tilde{\bar{\theta}}(\xi,p)$ where the tilde indicates Laplace transform with parameter p.

Finally, taking the inverse Laplace transform of $\tilde{\bar{\theta}}(\xi,p)$, followed by the inverse Fourier sine transform of the resulting expression for $\bar{\theta}(\xi,\beta)$, gives the solution for $\theta(\delta,\beta)$ when $B \geqslant 1/2$:

$$\theta_{B\geqslant 1/2} = 1-\frac{1}{\pi}\int_0^\infty I_1\,\mathrm{d}\xi \tag{6}$$

where

$$I_1 = \frac{\sin(\xi\delta)}{a\xi}\exp(-A\beta/2) \times[(2-B\xi^2)\sinh(a\beta)+2a\cosh(a\beta)] \tag{6a}$$

with $A = (2+B\xi^2)$ and $a = (A^2-4\xi^2)^{1/2}/2$. The integral in equation (6) must be evaluated numerically except for the Fourier case of $B = 1/2$ when the equation reduces to the widely-available Fourier solution (e.g. [8]). For numerical evaluation I_1 has the form 0/0 at $\xi = 0$, but L'Hôpital's Rule shows $I_1 = 2\delta$ there.

Equation (6) applies only to $B \geqslant 1/2$ because the argument of the square root in a becomes negative when $B < 1/2$ and, as described shortly, $\xi_1 < \xi < \xi_2$. However, when the argument is negative, factoring out $(-1)^{1/2} = i$ in every appearance of the square root in equation (6a) gives for $B < 1/2$:

$$\theta_{B<1/2} = 1-\frac{1}{\pi}\left\{\int_0^{\xi_1} I_1\,\mathrm{d}\xi+\int_{\xi_1}^{\xi_2} I_2\,\mathrm{d}\xi+\int_{\xi_2}^{\infty} I_1\,\mathrm{d}\xi\right\} \tag{7}$$

where

$$I_2 = \frac{\sin(\xi\delta)}{b\xi}\exp(-A\beta/2)[(2-B\xi^2)\sin(b\beta) +2b\cos(b\beta)] \tag{7a}$$

and $b = (4\xi^2-A^2)^{1/2}/2$. Equation (6a) leads to equation (7a) by using the identities: $\sinh(ib\beta) = i\sin(b\beta)$ and $\cosh(ib\beta) = \cos(b\beta)$.

The integration limits ξ_1 and ξ_2 in equation (7) are the values where $(A^2-4\xi^2)$, argument of the square root in a, changes sign from + to −, and − to +, respectively. Thus, the argument is negative for $\xi_1 < \xi < \xi_2$. Consequently, the three integrals in equation (7) result from dividing the entire integration range of ξ into subintervals corresponding to positive and negative arguments of the square root in a, where I_2 corresponds to a negative argument. These values of ξ_1 and ξ_2 are found by solving $(A^2-4\xi^2) = 0$ and rejecting negative roots since $\xi \geqslant 0$:

$$\xi_1 = [2(1-B)-2(1-2B)^{1/2}]^{1/2}/B \tag{8a}$$

$$\xi_2 = [2(1-B)+2(1-2B)^{1/2}]^{1/2}/B. \tag{8b}$$

For numerical evaluation, the integrands in equation (7) have the form 0/0 at ξ_1 and ξ_2, but L'Hôpital's Rule shows

$$I_1 \text{ and } I_2 = \frac{\sin(\xi\delta)}{\xi}\exp(-A\beta/2)[2+\beta(2-B\xi^2)] \tag{9}$$

with ξ representing ξ_1 or ξ_2. For $B = 0$, however, equation (9) pertains only to ξ_1, since $\xi_1 = 1$ and $\xi_2 = \infty$ for this value of B.

2.3. *Alternative to previous solutions*

In contrast to the solution procedure just summarized, Reference [6] solves the problem of equations (4)–(4d) using the Laplace transform followed by contour integration to obtain the inverse transform. Subsequent discussion in [6] involves only $B > 1/2$.

Although not reported in [6], its solution is restricted to $B \geqslant 1/2$ because of the contour integration. Reference [9] identifies this restriction in the analogous context of a viscoelastic liquid, where a solution corresponding to $B < 1/2$ using a different contour is obtained for the analogous viscoelastic problem. In this analogy a half-space of liquid, initially quiescent, is subjected to a step increase of velocity along its surface.

While all values of B are accounted for by the previous solutions in [6] and [9], the new solution given here by equations (6) and (7) can be a convenient alternative. For instance, equation (6) requires only one numerical integration while the solution in [6] requires two numerical integrations.

3. CONSTANT SURFACE HEAT FLUX

3.1. *Formulation*

The semi-infinite slab occupies $x > 0$ and is initially at T_i while heat flux throughout the slab is initially zero. At $t = 0^+$ the constant heat flux q_0 is applied suddenly and uniformly along the surface $x = 0$, so surface temperature increases for $t > 0$. A solution for $T(x,t)$ in the slab could be derived by solving equation (3) using equation (1) to specify the boundary condition of constant heat flux along $x = 0$. Unfortunately, this boundary condition would be a complex relation between q_0 and $T(0,t)$.

To avoid this complexity an alternative approach is taken here. This approach takes advantage of the analogy between governing equations for $T(x,t)$ and $q(x,t)$ noted previously with equation (3), and the solution just obtained for the problem with constant surface temperature: First, the solution for $q(x,t)$ in the slab with constant surface heat flux is obtained by analogy with the previous problem of constant surface temperature. Then, the desired solution for $T(x,t)$ with constant surface heat flux is obtained by inserting this $q(x,t)$ into equation (2) (the energy equation) and integrating for temperature. Hence, by defining the dimensionless heat flux $\psi(\delta,\beta) = q(x,t)/q_0$ for the slab with constant surface heat flux, equations (4)–(4d) become the analogous problem for ψ after replacing θ by ψ.

Solution

Equations (6) and (7) give the solution for ψ in the slab with constant surface heat flux after replacing θ by ψ. Then, defining the dimensionless temperature $\varphi(\delta,\beta) = (T-T_i)k/[q_0(\alpha\tau_q)^{1/2}]$ for this slab and noting $\varphi(\delta,0) = 0$, integrating the dimensionless form of equation (2) gives

$$\varphi = -\int_0^\beta\left(\frac{\partial\psi}{\partial\delta}\right)\mathrm{d}u \tag{10}$$

where u is a dummy integration variable for β. Next, inserting equation (6) for ψ into equation (10), evaluating the δ-derivative with Liebnitz's Rule, then interchanging the orders of integration and performing the β-integration gives for $B \geqslant 1/2$:

$$\varphi_{B\geqslant 1/2} = \frac{1}{\pi}\int_0^\infty I_3\,\mathrm{d}\xi \tag{11}$$

where

$$I_3 = \frac{\cos(\xi\delta)}{a\xi^2}[4a-\exp(-A\beta/2) \times\{[4+2\xi^2(B-1)]\sinh(a\beta)+4a\cosh(a\beta)\}]. \tag{11a}$$

Equation (11) reduces to the Fourier solution [8] for $B = 1/2$. For other values of B the integral in the equation must be evaluated numerically using $I_3 = 2\beta$ at $\xi = 0$, found with L'Hôpital's Rule.

For $B < 1/2$, inserting equation (7) for ψ into equation (10) leads to

$$\varphi_{B<1/2} = \frac{1}{\pi}\left\{\int_0^{\xi_1} I_3\,\mathrm{d}\xi + \int_{\xi_1}^{\xi_2} I_4\,\mathrm{d}\xi + \int_{\xi_2}^{\infty} I_3\,\mathrm{d}\xi\right\} \tag{12}$$

where

$$I_4 = \frac{\cos(\xi\delta)}{b\xi^2}[4b - \exp(-A\beta/2)$$

$$\times \{[4+2\xi^2(B-1)]\sin(b\beta) + 4b\cos(b\beta)\}]. \tag{12a}$$

The values of ξ_1 and ξ_2 are still given by equations (8a) and (8b). Also, since I_3 and I_4 have the form 0/0 at ξ_1 and ξ_2, L'Hôpital's Rule shows

$$I_3 \quad \text{and} \quad I_4 = 2\frac{\cos(\xi\delta)}{\xi^2}\{2 - \exp(-A\beta/2)$$

$$\times (2+\beta[2+\xi^2(B-1)])\} \tag{13}$$

where ξ represents ξ_1 or ξ_2. For $B = 0$, however, equation (13) pertains to only ξ_1, since $\xi_1 = 1$ and $\xi_2 = \infty$ for this value of B.

3.2. *Illustration of results*

The behavior of φ is illustrated here treating B as a parameter, by varying τ_T while holding τ_q constant. The hyperbolic case $B = 0$ is included for comparison since experiments with processed meat show good agreement with hyperbolic predictions [10]. For the hyperbolic case, a jump in surface temperature occurs immediately upon application of the heat flux, followed by propagation of a thermal wave into the slab [11]. In terms of the dimensionless variables used here this jump is $\varphi_{\text{jump}} = 1$ at time $\beta = 0^+$. Also, the transient location of the thermal wave-front is $\delta_w = \beta$, obtained by converting $x_w = c_w t$ into dimensionless variables along with $\tau_q = \alpha/c_w^2$ [2], where c_w is the thermal wave speed.

The numerical integrations for equations (11) and (12) were performed with Simpson's Rule. Convergence studies showed upper limits of 1×10^5 to simulate infinity, along with sufficient subdivisions of integration intervals, gave accurate integrations since higher limits showed $<1\%$ change in φ. Also, results obtained with equations (11) and (12) were tested by checking for agreement with independent solutions for the Fourier ($B = 1/2$) [8] and hyperbolic ($B = 0$) [11] cases.

Figure 1 shows surface temperature of the slab vs time for B ranging from 0 to 1/2. The hyperbolic case ($B = 0$) shows its jump in surface temperature at $\beta = 0^+$. Because $\tau_T = 0$ for the hyperbolic case, the jump results from delayed conduction into the slab caused by the thermal inertia effect of τ_q. Consequently, this delay initially confines heat to the surface. In comparison, for the DPL case ($B > 0$) there is no temperature jump because the added effect of $\tau_T \neq 0$ promotes conduction into the slab. However, for sufficiently small B (e.g. 0.2) the rapid increase in surface temperature shortly after $\beta = 0$ can closely approximate the temperature jump of the hyperbolic case.

The figure also shows that increasing B reduces the initial rate of increase in surface temperature, because τ_T promotes conduction into the slab rather than confining heat to the surface. As time increases, however, all temperatures converge to the Fourier case ($B = 1/2$).

Next, Fig. 2 shows surface temperature vs time for $1/2 \leqslant B \leqslant 100$. The figure further illustrates that increasing B reduces the initial rate of temperature increase. Although not shown here, all temperatures eventually converge to the Fourier case. Also, it is interesting to note the correspondence between the dimensionless variables of the figure and actual variables. For instance, using $\tau_q \approx 0.7 \times 10^{-12}$ s for gold [2], the dimensionless time $\beta = 50.0$ corresponds to the actual time of 7.0×10^{-11} s. This time is comparable to, for example, time scales involved with microtime laser heating of metals [7].

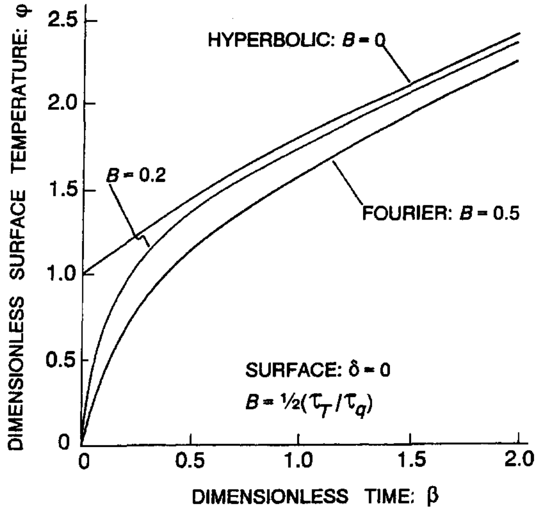

Fig. 1. Surface temperature vs time for semi-infinite slab with constant surface heat flux: $0 \leqslant B \leqslant 1/2$.

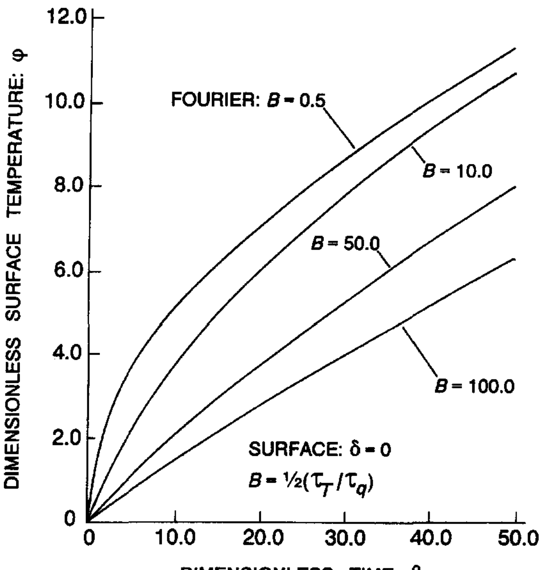

Fig. 2. Surface temperature vs time for semi-infinite slab with constant surface heat flux: $1/2 \leqslant B \leqslant 100$.

Figure 3 shows the effect of B on internal temperature profiles at the representative time $\beta = 0.1$. The hyperbolic case ($B = 0$) shows its thermal wave with wave-front located at $\delta_w = 0.1$. The wave-front is the location where temperature first begins to increase in overcoming the thermal inertia effect of τ_q. As B increases for the DPL case, however, the wave-front is progressively smoothed by the increasing effect of τ_T promoting conduction into the slab. However, for sufficiently small B (e.g. 0.01) the DPL profile can closely approximate the hyperbolic profile. This smoothing of profiles and promotion of conduction is consistent with inde-

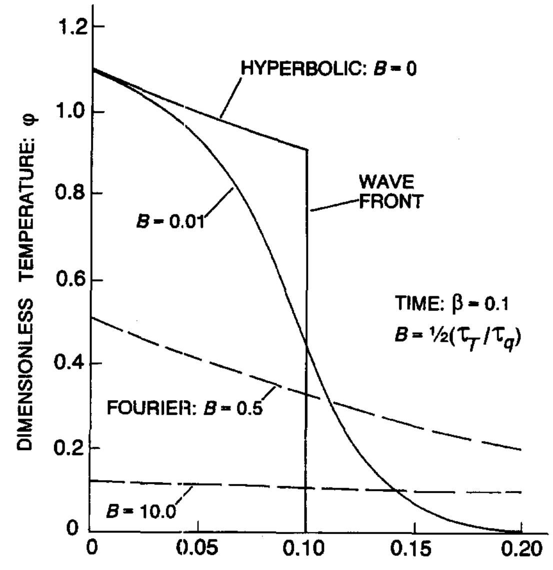

Fig. 3. Effect of B on internal temperature profiles for semi-infinite slab with constant surface heat flux.

pendent DPL results for the problem of constant surface temperature [12].

Finally, Fig. 3 shows that increasing B "flattens" the temperature profiles because each profile reflects the same amount of energy provided by the surface heat flux. Thus, as B increases and profiles extend deeper into the slab, they must flatten to reflect the same energy content. Also, some effect of conduction always extends to $\delta \to \infty$ for $B > 0$ since equation (3) is parabolic [13]. (However, higher order DPL models [1, 3] can predict non-Fourier thermal waves where conduction does not extend to $\delta \to \infty$.) All profiles converge to the Fourier case ($B = 1/2$) as time increases.

In conclusion, Figs. 1–3 show the DPL model can predict surface and internal temperatures that are very different from predictions based on Fourier's Law. Further, DPL predictions can closely approximate those of hyperbolic conduction for sufficiently small values of thermal lag ratio τ_T/τ_q. All predictions converge to those of Fourier's Law at sufficiently large time.

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APPENDIX

Here, equation numbers followed by R refer to those in Reference [1] while A denotes equations appearing only in this Appendix. Other equation numbers and all reference numbers refer to those in the main body of this Note.

The solution for a semi-infinite slab with constant surface temperature given by equation (2.61R) on p. 47 of [1] is correct, but pertains to a different problem than the solution given by equations (6) and (7) in this Note. This difference stems from the statements of boundary conditions at the surface of the slab. Specifically, equation (2.27R) on p. 36 of [1] states $\theta(0,\beta) = 1$ for $\beta > 0$, as implied by equation (2.21R) on p. 35. However, equation (4c) of this Note states $\theta(0,\beta) = H(\beta)$, where $H(\beta) = 0$ for $\beta < 0$ and $H(\beta) = 1$ for $\beta > 0$ [14]. The consequence of these statements becomes clear by examining an intermediate stage of the analysis from this Note, then contrasting this stage to the analysis in [1].

For the slab with constant surface temperature in this Note, taking the Fourier sine transform of each term in equation (4) to eliminate δ-derivatives gives

$$\frac{\mathrm{d}^2\bar{\theta}}{\mathrm{d}\beta^2}+2\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\beta} = (2/\pi)^{1/2}\xi H(\beta)-\xi^2\bar{\theta}+B\frac{\mathrm{d}}{\mathrm{d}\beta}[(2/\pi)^{1/2}\xi H(\beta)-\xi^2\bar{\theta}] \tag{A1}$$

where $H(\beta)$ enters equation (A1) through the transform of terms in equation (4) involving δ-derivatives. Then, taking the β-derivative shown in the last term of equation (A1), and noting $\mathrm{d}H(\beta)/\mathrm{d}\beta = \Delta(\beta)$ where Δ is the delta (unit impulse) function [14], leads to

$$\frac{\mathrm{d}^2\bar{\theta}}{\mathrm{d}\beta^2}+(2+B\xi^2)\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\beta}+\xi^2\bar{\theta} = (2/\pi)^{1/2}\xi[H(\beta)+B\Delta(\beta)]. \tag{A2}$$

Solving equation (A2) with its initial conditions as described previously in this Note results in the solution given by equations (6) and (7). This solution reduces to the Fourier case for $B = 1/2$, as it should. Hence, equations (6) and (7) serve as the starting point in this Note for solving the problem of the semi-infinite slab with constant heat flux at its surface.

In contrast to using $H(\beta)$ for the slab with constant surface temperature as in this Note, using the boundary condition $\theta(0,\beta) = 1$ from Reference [1] replaces each occurrence of $H(\beta)$ by 1 in equation (A1). Thus, $B\Delta(\beta)$ does not arise in the analysis on p. 47 of [1] even though that analysis also uses the Fourier sine transform. Further, equation (A2) reverts to equation (2.57R), leading to the solution given by equation (2.61R). This solution is not adopted for this Note because it does not reduce to the Fourier case for $B = 1/2$. The solution, however, does not affect the remainder of [1] since it is not used in further developments there. Also, for $B = 0$ the solutions in this Note and [1] become identical because $B\Delta(\beta)$ vanishes from equation (A2).

Consequently, the solutions in this Note and [1] are different because the Fourier sine transform introduces the boundary condition into the transform of equation (4) where it is then subjected to the β-derivative. Hence, while the two statements of boundary conditions are the same in practical terms, the formal statement $\theta(0,\beta) = H(\beta)$ "captures" the suddenly-raised surface temperature at $\beta = 0^+$ because its derivative gives the delta function. However, $\theta(0,\beta) = 1$ does not "capture" this temperature change because its derivative is zero. Instead, $\theta(0,\beta) = 1$ effectively corresponds in this instance to the different problem of surface temperature held at the value of 1 for all time, with conduction into the solid beginning at $\beta = 0^+$.

Finally, using the Laplace transform to eliminate β-derivatives as the first step of analysis, as done in [6] and on p. 37 of [1], renders the distinction between $H(\beta)$ and 1 at the surface unimportant because their β-derivatives do not arise in subsequent steps. In general, however, the solution of future DPL problems using the approach taken in this Note would require formal statements of time-dependent boundary conditions.